# Linear Projections of Smooth Projective Threefolds (Hodge theory and algebraic geometry)

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# Linear Projections of Smooth Projective Threefolds

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**Abstract.** In [17] and [18] we have proved formulas which give the Chern numbers of the normalization $X$ of a hypersurface with ordinary singularities $\overline{X}$ in $P^4(\mathbb{C})$. In this article, in order to obtain concrete examples of hypersurfaces with ordinary singularities in $P^4(\mathbb{C})$, we embed smooth rational threefolds such as $P^1(\mathbb{C}) \times P^2(\mathbb{C})$, $P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ and $P^3(\mathbb{C})$ into higher dimensional projective spaces by the use of monomials, and project them to 4-dimensional linear subspaces of the projective spaces. We count numerical invariants of the hypersurfaces with ordinary singularities in $P^4(\mathbb{C})$, obtained in this way, and calculate concrete equations of the hypersurfaces in some cases by the aid of computer. These are expected to be useful to see that our formulas for the Chern numbers certainly hold.

## 1 Singularities of the image of a smooth projective threefold by a generic linear projection

Throughout this article we work over the complex number field $\mathbb{C}$. Let $X$ be an $n$-dimensional smooth subvariety of $P^N(\mathbb{C})$, and $\Lambda$ an (N-m-1)-dimensional linear subspace of $P^N(\mathbb{C})$, $Y$ an $m$-dimensional linear subspace of $P^N(\mathbb{C})$ such that $\Lambda$ and $Y$ are situated in general position. We assume that $X \cap \Lambda = \emptyset$, and so

$$(N-m-1)+n < N \Longleftrightarrow n < m+1.$$

**Definition 1.1.** For $X$, $\Lambda$ and $Y$ as above, we define the *linear projection* $\pi_\Lambda : X \to Y$ of $X$ from $\Lambda$ to $Y$ by

$$\pi_\Lambda(x) := L(x, \Lambda) \cap Y \qquad (x \in X),$$

where $L(x, \Lambda)$ denotes the $(N-m)$-dimensional linear subspace of $P^N(\mathbb{C})$ generated by $x$ and $\Lambda$.

We denote by $\mathbb{G}(N\text{-}m\text{-}1, N)$ the Grassmann variety of (N-m-1)-linear subspaces of $P^N(\mathbb{C})$. We regard $\Lambda$ as an element of $\mathbb{G}(N\text{-}m\text{-}1, N)$ and vary it.

If there is a dense open subset $U$ of $\mathbb{G}(N\text{-}m\text{-}1, N)$ such that a linear projection $\pi_\Lambda$ for any $\Lambda \in U$ has a "good" property , we say that a "generic" linear projection $\pi_\Lambda$ has the "good" property. We are interested in the singularities of the image

$$\pi_\Lambda(X) \subset Y = P^m(\mathbb{C})$$

of $X \subset P^N(\mathbb{C})$ by a "generic" linear projection $\pi_\Lambda$ for the case $n < m$, especially, the case $n = m-1$, that is, the case where $\pi_\Lambda(X)$ is a *hypersurface*.

**Proposition 1.1.** *When $n = 3$, for a "generic" $\Lambda \in \mathbb{G}(N\text{-}5, N)$, the local analytic equations of $\pi_\Lambda(X)$ are given by one of the following:*

$$\begin{cases} \text{(i) } w = 0 & \text{(simple point)} \\ \text{(ii) } zw = 0 & \text{(ordinary double point)} \\ \text{(iii) } yzw = 0 & \text{(ordinary triple point)} \\ \text{(iv) } xyzw = 0 & \text{(ordinary quadruple point)} \\ \text{(v) } xy^2 - z^2 = 0 & \text{(cuspidal point)} \\ \text{(vi) } w(xy^2 - z^2) = 0 & \text{(stationary point),} \end{cases}$$

*where $(x, y, z, w)$ is the coordinate on $\mathbb{C}^4$.*

**Definition 1.2.** The singularity listed in the proposition above are called *ordinary singularities* of dimension 3.

The statement of Proposition 1.1 can be found in Roth's book "Algebraic Threefold" (Springer-Verlag, Berlin, 1955). We can prove this by the use of an analytic version of the theory of "stable map" thanks to Mather. Originally, "stable map" is a notion in $C^\infty$ category, and is global one, though the global notion of "stable map" is invalid in complex analytic category. In complex analytic category, instead, we use the notion of "locally stable holomorphic map", which is defined as follows: Let $f : M \to N$ be a holomorphic map between complex manifolds, and $S$ a finite subset of $M$. We denote by $f : (M, S) \to (N, f(S))$ the *multi-germ of a holomorphic map* $f$ at $S$.

**Definition 1.3.** A multi-germ of a holomorphic map $f : (M, S) \to (N, f(S))$ is defined to be *stable* if any deformation (= parametrized unfolding) of it is *trivial.*

**Definition 1.4.** A holomorphic map between complex manifolds $f : M \to N$ is defined to be *locally stable* if for any finite subset $S$ of $M$, the multi-germ of a holomorphic map $f : (M, S) \to (N, f(S))$ is stable.

With these notation and terminology, we have:

**Theorem 1.2.** ([10]) *Let $X$, $\Lambda$ and $Y$ be the same as in Definition 1.1. If $(n, m)$ belongs to the so-called "nice range", then there exists a dense open subset $U$ of $\mathbb{G}(N\text{-}m\text{-}1, N)$ such that, for any $\Lambda \in U$, the linear projection $\pi_\Lambda : X \to Y$ of $X$ from $\Lambda$ to $Y$ is a locally stable holomorphic map.*

Here we do not explain what *"nice range"* is, but we only mention that in the case $m = n+1$, $(n, m)$ belongs to the *"nice range"* if and only if $n \leq 14$. From this theorem we can derive the following:

**Proposition 1.3.** *Let X be a smooth algebraic threefold embedded in $P^N(\mathbb{C})$ $(N \geq 5)$, and $\Lambda$ an $(N\text{-}5)$-dimensional linear subspace of $P^N(\mathbb{C})$, $Y$ a 4-dimensional linear subspace of $P^N(\mathbb{C})$ such that $\Lambda$ and $Y$ are situated in general position. Then there exists a dense open subset $U$ of $\mathbb{G}(N\text{-}m\text{-}1, N)$ such that, for any $\Lambda \in U$, the image of $X$ in $Y$ by the linear projection $\pi_\Lambda : X \to Y$ from $\Lambda$ to $Y$ is a hypersurface with ordinary singularities in $Y$.*

Roughly speaking, the proof of Proposition 1.3 proceeds as follows: First note that the pair of integers $(3,4)$ surely belongs to the so-called "nice range". Generally, stable holomorphic map germs at a point are classified by the $\mathbb{C}$-algebra

$$R_f := \mathcal{O}_{X,p}/f^*\mathfrak{m}_{Y,f(p)} \qquad (\mathfrak{m}_{Y,f(p)} : \text{the maximal ideal of } \mathcal{O}_{Y,f(p)})$$

associated to $f : (X,p) \to (Y, f(p))$. In the case where $\dim X = 3$ and $\dim Y = 4$, the $\mathbb{C}$-algebra associated to a stable holomorphic germ at a point is restricted to one of the following:

$$A_0 = \mathbb{C}[[x]]/(x), \quad A_1 = \mathbb{C}[[x]]/(x^2).$$

$R_f \simeq A_0$ is the case when $f$ is non-degenerated at $x$, i.e., the Jacobian $df$ of $f$ has maximal rank at $x$. The *normal form* of the stable map germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ with $R_f \simeq A_1$ is given by

$$\begin{cases} y_1 \circ f = x_1 \\ y_2 \circ f = x_2 \\ y_3 \circ f = x_3^2 \\ y_4 \circ f = x_1 x_3, \end{cases}$$

and if we define

$$C(A_1) := \left\{ x \in \mathbb{C}^3 \mid R_{f_x} \simeq A_1 \right\}$$

where $f_x$ denotes the map germ of $f$ at $x \in \mathbb{C}^3$, then

$$C(A_1) : x_1 = x_3 = 0.$$

The equation of $f(\mathbb{C}^3) \subset \mathbb{C}^4$ at $0$ is given by

$$y_3 y_1^2 - y_4^2 = 0,$$

which is the so-called *Whitney umbrella*, or *cuspidal point*, or *pinch point*. By this and the fact that a locally satable holomorphic map is a *Thom-Boardman map satisfying condition NC* (*normal crossing*), we have the proposition above (For details, see [14]). For the precise definition of a *Thom-Boardman map satisfying condition NC* (*normal crossing*), see [4].

# 2 Chern numbers of the normalization of a hypersurface with ordinary singularities in $P^4(\mathbb{C})$

Throughout §§2, 3, we fix the notation as follows:

$Y := P^4(\mathbb{C})$ : the complex projective 4-space,  
$\overline{X}$ : an algebraic threefold with ordinary singularities in $Y$,  
$\overline{J}$ : the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$,  
$\overline{D}$ : the singular locus of $\overline{X}$,  
$\overline{T}$ : the triple point locus of $\overline{X}$, which is equal to the singular locus of $\overline{D}$,  
$\overline{C}$ : the cuspidal point locus of $\overline{X}$, precisely, its closure, since we always consider $\overline{C}$ contains the stationary points,  
$\Sigma\overline{q}$ : the quadruple point locus of $\overline{X}$,

$\overline{\Sigma s}$ : the stationary point locus of $\overline{X}$,
$n_{\overline{X}} : X \to \overline{X}$ : the normalization of $\overline{X}$,
$f : X \to Y$ : the composite of the normaliztion map $n_{\overline{X}}$ and the inclusion $\iota : \overline{X} \hookrightarrow Y$,
$J$ : the scheme-theoretic inverse of $\overline{J}$ by $f$,
$D$, $T$, $C$ and $\Sigma s$ : the inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\overline{\Sigma s}$ by $f$, respectively.

We put

$$n := \deg \overline{X} \text{ (the degree of } \overline{X} \text{ in } P^4(\mathbb{C})), \; m := \deg \overline{D}, \; t := \deg \overline{T}, \; \gamma := \deg \overline{C}.$$

Note that $\overline{T}$ and $\overline{C}$ are smooth curves, intersecting transversely at $\overline{\Sigma s}$, and that the normalization $X$ of $\overline{X}$ is also smooth. Calculating by the use of local coordinates, we can easily see the following:

(i) $J$ contains $D$, and the *residual scheme* to $D$ in $J$ is the reduced scheme $C$, i.e., $\mathcal{I}_J = \mathcal{I}_D \otimes_{\mathcal{I}_X} \mathcal{I}_C$, where $\mathcal{I}_J$, $\mathcal{I}_D$, $\mathcal{I}_C$ are the ideal sheaves of $J$, $D$ and $C$, respectively (cf. [3], Definition 9.2.1, p.160);

(ii) $D$ is a surface with ordinary singularities, whose singular locus is $T$,

(iii) $D$ is the *double point* locus of the map $f : X \to Y$, i.e., the closure of $\{q \in X \mid \#f^{-1}(f(q)) \geq 2\}$ ;

(iv) the map $f_{|D} : D \to \overline{D}$ is generically two to one, simply ramified at $C$;

(v) the map $f_{|T} : T \to \overline{T}$ is generically three to one, simply ramified at $\Sigma s$.

Concerning the Euler number of $X$, denoted by $\chi(X)$, we have the following:

**Proposition 2.1.** ([16], Proposition 2.3)

$$\chi(X) = n(2n^2 - 7n + 9) - 2(3n - 7)m + 6t - 4\gamma - c \tag{2.1}$$

*where $c$ denotes the class of $\overline{X}$, i.e., the number of hyperplanes being tangent to $X$ at a point and passing through a fixed generic 2-linear subspace of $P^4(\mathbb{C})$.*

To prove the proposition above we use a Lefschetz pencil $\overline{\mathcal{L}} = \bigcup_{\lambda \in P^1} \overline{X}_\lambda$ on $\overline{X}$, consisting of hyperplane sections of $\overline{X}$. We denote by $\overline{B}$ the base point locus of $\overline{\mathcal{L}}$, which is an irreducible curve of degree $n$ with $m$ nodes on $\overline{X}$. Let $\sigma : \tilde{X} \to X$ be the blowing-up along $n_{\overline{X}}^{-1}(\overline{B})$, and let $\tilde{\mathcal{L}} = \bigcup_{\lambda \in P} \tilde{X}_\lambda$ be the pull-back of $\overline{\mathcal{L}}$ to $\tilde{X}$ by $n_{\overline{X}} \circ \sigma$. Then $\tilde{\mathcal{L}}$ gives a fibering of $\tilde{X}$, whose fiber is a smooth surface except over finite points $\lambda_1, \cdots, \lambda_c$ of $P^1$. Every singular fiber over $\lambda_i$ $(1 \leq i \leq c)$ is a surface with only one isolated ordinary double point. The Euler number of a general fiber $\tilde{X}_\lambda$ is given by

$$\chi(\tilde{X}_\lambda) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma,$$

wnich is a classical formula for surfaces with ordinary sungularites. From thses facts, (2.1) follows.

The formulas for the Chern numbers of $X$ are as follows:

**Theorem 2.2.**

$$\int_X c_3 = \chi(X) = -n(n^3 - 5n^2 + 10n - 10) + (4n^2 - 15n - 2m + 20)m - (4n - 15)t$$
$$+ (n + 10)\gamma + 5\deg[K_X \cdot C] - \#\overline{\Sigma s} + 2\chi(\overline{C}, \mathcal{O}_{\overline{C}}) + 4\#\overline{\Sigma q}.$$

$$\int_X c_1^3 = -n(n-5)^3 + 6(n-5)^2 m - 3(n-5)(nm + 3t - \gamma)$$
$$+ (n^2 - 2m)m + 5nt - (2n - 5)\gamma + \deg[K_X \cdot C] - \#\overline{\Sigma s} + 4\#\overline{\Sigma q}.$$

$$\int_X c_1 c_2 = -24\chi(X, K_X) = -24\chi(Y, \mathcal{O}_Y([(n-5)H] - \overline{D})) + 24$$
$$= -(n-4)(n-3)(n-2)(n-1) + 24\chi(\overline{D}, \mathcal{O}_{\overline{D}}(n-5)) + 24,$$

*where $K_X$ is a canonical divisor of $X$.*

**Remark 2.1.** As pointed out in [18], the formulas for $\int_X c_3$ and $\int_X c_1^3$ in [17] are false. This is because the diagram

$$\begin{tikzcd} C \arrow[r, "\iota"] \arrow[d, "f|_C"'] & X \arrow[d, "f"] \\ \overline{C} \arrow[r, "\bar{\iota}"'] & Y. \end{tikzcd}$$

is not Cartesian, since $[f^{-1}(\overline{C})] = 2[C]$, and so we cannot apply the *excess intersection formula* (cf. [3], Theorem 6.3, p.102) to calculate $f^*[\overline{C}]$. Hence, the identity

$$f^*[\overline{C}] = f^*[\overline{X}] \cdot [C] - [D \cdot C],$$

on page 299 in [17] is incorrect, and the second identity at (3.26) on the same page in [17] must be replaced by

$$[D \cdot C] = f^*[\overline{X} + K_Y] \cdot [C] - [K_X \cdot C],$$

which follows from the *double point formula* $[D] = f^*[\overline{X} + K_Y] - [K_X]$, where $K_X$ and $K_Y$ are canonical divisors of X and Y, respectively.

The most hard part of Theroem 2.2 is the first formula for the Euler number. The class c is nothing but the degree of the top *polar class* of $\overline{X}$. Thanks to Piene's formula in [11], calculating the *Segre classes* of the singular subscheme $\overline{J}$ of $\overline{X}$, we have

$$c = (n-1)^3 n - (4n^2 - 9n - 2m + 6)m + (4n-9)t - (n+14)\gamma - 5\deg[K_X \cdot C] + \#\Sigma\overline{s} - 2\chi(\overline{C}, \mathcal{O}_{\overline{C}}) - 4\#\Sigma\overline{q}.$$

For the precise definitions of *polar class* and *Segre class*, see [11] or [3].

# 3 Examples

The following is an example of a 2-dimensional hypersurface with ordinary singularities in $P^3(\mathbb{C})$, named *Steiner surface*:

$$(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0,$$

where $[x : y : z : w]$ is the homogeneous coordinate on $P^3(\mathbb{C})$. Its singular locus consists of the three lines $\Lambda_0$, $\Lambda_1$ and $\Lambda_2$ defined by $x = y = 0$, $y = z = 0$ and $z = x = 0$, respectively, which we call the *double curves* of it. The Steiner surface has one ordinary triple point $[0 : 0 : 0 : 1]$, six ordinary cuspidal points $[1 : 0 : 0 : \sqrt{2}]$, $[1 : 0 : 0 : -\sqrt{2}]$, $[0 : 1 : 0 : \sqrt{2}]$, $[0 : 1 : 0 : -\sqrt{2}]$, $[0 : 0 : 1 : \sqrt{2}]$, $[0 : 0 : 1 : -\sqrt{2}]$, two of which lie on each of the line $\Lambda_i$, and no quadruple point. The Steiner surface is obtained as the image of $P^3(\mathbb{C})$ by the composite of the quadratic Veronese map (embedding)

$$\begin{aligned} \nu_2 : [\xi_0 : \xi_1 : \xi_2] \quad &\in \quad P^2(\mathbb{C}) \\ &\to \quad [\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_1\xi_2] = [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \in P^5(\mathbb{C}) \end{aligned}$$

and the linear projection

$$\begin{aligned} \pi_L : (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \quad &\in \quad P^5(\mathbb{C}) \\ &\to \quad (y_0 : y_1 : y_2 : -(x_0 + x_1 + x_2)) = (x : y : z : w) \in P^3(\mathbb{C}) \end{aligned}$$

The center of the linear projection $\pi_L$ is the line

$$L : y_0 = y_1 = y_2 = x_0 + x_1 + x_2 = 0.$$

In what follows we try to find out similar examples in 3-dimensional case. First, we recall the formulas which express *multiple-point cycle classes* and *ramification point cycle classes* in terms of invariants of X and Y for an appropriately generic morphism $f : X \to Y$ between smooth algebraic varieties with

$\dim X < \dim Y$, where *cycle classes* mean equivalence classes in the ring $A.X$ of algebraic cycles on X modulo rational equivalence. We set

$$M_r := \left\{ x \in X \mid \text{there exist } r \text{ distinct points (possibly infinitely near each other) in } f^{\cdot\, 1} f_\bullet x_\bullet \right\},$$

and call it the *r-fold point locus of* f. $M_r$ has naturally the structure of a reduced subscheme. We denote by $[M_r]$ the element of $A.X$ determined by $M_r$. We set $n = \dim X$, $m = \dim Y$ $(n < m)$, and

$$R := \left\{ x \in X \mid \operatorname{rank}(df)_x \leq n-1 \right\},$$

where df is the Jacobian map of f. R is called the *ramification locus* of f, or the *singular locus* of f. R has naturally subscheme structure; it is defined by the ideal generated by the n-mionors of $df : \tau_X \to f^*\tau_Y$, where $\tau_X$ and $\tau_Y$ denote the tangent bundles of X and Y, respectively. We denote by [R] the element of A.X determined by R.

**Theorem 3.1.** *Let* X *be a smooth algebraic threefold embedded in* $P^N(\mathbb{C})$ $(N \geq 5)$*,* Y *a* 4*-dimensional linear subspace of* $P^N(\mathbb{C})$*, and* $\pi_\Lambda : X \to Y$ *the linear projection of* X *from an* $(N-5)$*-dimensional linear subspace* $\Lambda$ *of* $P^N(\mathbb{C})$ *to* Y*. We denote by* $\overline{X}$ *the image of* X *by* $\pi_\Lambda$*. If* $\pi_\Lambda$ *is generic, that is, if* $\Lambda$ *corresponds to a point of a suitable dense open subset of the Grassmann varity* $\mathbb{G}(N\text{-}5, N)$ *, then* $M_i$ *is empty for* $i \geq 5$ *and*

$$\dim M_i = 4 - i \qquad (2 \leq i \leq 4).$$

*Furthermore, under the same assumption, we have:*

$$[M_2] = \pi_\Lambda^*[\overline{X} + K_Y] - K_X,$$

$$[M_3] = \frac{1}{2!}\left\{ [M_2]^2 - [M_2] \cdot \pi_\Lambda^* c_1(Y) + 2c_2(\nu) + \pi_\Lambda^*\pi_{\Lambda *} c_1(X) - c_1(\nu)c_1(X) \right\},$$

$$[M_4] = \frac{1}{3!}\left\{ \pi_\Lambda^*\pi_{\Lambda *} 2![M_3] - 3c_1(\nu) \cdot (2![M_3]) + 6c_2(\nu)[M_2] - 6c_1(\nu)c_2(\nu) - 12c_3(\nu) \right\},$$

*where* $\nu := \pi_\Lambda^*\tau_Y - \tau_X$ *is an element of* $K(X)$*, called the vertual normal sheaf of* $\pi_\Lambda$*.*

The above theorem is a conclusion derived from *multiple-point formulas* due to S. L. Kleimen ([6], [7]).

**Theorem 3.2.** *With the same notation and under the same assumption as in Theorem 3.1,* R *is a smooth curve (possibly reducible), and*

$$[R] = c_2(\nu).$$

The fact that R is smooth follows from that $\pi_\Lambda$ is a Thom-Boardman map. The last identity in the theorem above is a conclusion derived from the *Porteous formula* ([12]).

In the subsequence, we denote by $H_{P^n}$ a generic hyperplane in $P^n(\mathbb{C})$, and by $H^i_{P^n}$ the intersection of i hyperplanes in general position in $P^n(\mathbb{C})$.

**Example 3.1** (Generic projection of Segre threefold): Let $s : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^5(\mathbb{C})$ be the map defined by

$$\begin{array}{rcl} [s_0 : s_1] \times [t_0 : t_1 : t_2] & \in & P^1(\mathbb{C}) \times P^2(\mathbb{C}) \\ & \to & [s_0t_0 : s_0t_1 : s_0t_2 : s_1t_0 : s_1t_1 : s_1t_2] = [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \in P^5(\mathbb{C}) \end{array}$$

i.e., the *Segre map* from $P^1(\mathbb{C}) \times P^2(\mathbb{C})$ to $P^5(\mathbb{C})$. We set

$$\Sigma_{1,2} := s(P^1(\mathbb{C}) \times P^2(\mathbb{C})),$$

which is called *Segre threefold*. It is a *rational normal scroll*, and as such is denoted by $X_{1,1,1}$, whose meaning is as follows: We take three points $p_0, p_1, p_2$ in general position in $P^2(\mathbb{C})$,and set

$$L_0 := s(P^1(\mathbb{C}) \times p_0),$$

$$L_1 := s(P^1(\mathbb{C}) \times p_1)$$
$$L_2 := s(P^1(\mathbb{C}) \times p_1).$$

These are three lines in general position in $P^5(\mathbb{C})$. We denote the natural isomorphisms

$$\varphi_i : L_0 \to L_i \quad (i = 1, 2).$$

Then $\Sigma_{1,2}$ is described as

$$\Sigma_{1,2} = \bigcup_{p \in L_0} \overline{p, \varphi_1(p), \varphi_2(p)},$$

where $\overline{p, \varphi_1(p), \varphi_2(p)}$ denotes the 2-dimensional linear subspace of $P^5(\mathbb{C})$, generated by $p, \varphi_1(p)$ and $\varphi_2(p)$.

**Proposition 3.3.** *We denote by $\overline{\Sigma_{1,2}}$ the image of $\Sigma_{1,2}$ by a generic linear projection from a point $p \in P^5(\mathbb{C})$ to $P^4(\mathbb{C})$. Then:*

$$\deg \overline{\Sigma_{1,2}} = 3.$$

**Proof:** By the definition of $s : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^5(\mathbb{C})$,

$$s^*[\Sigma_{1,2} \cap H_{P^5}] = [H_{P^1} \times P^2] + [P^1 \times H_{P^2}].$$

Hence

$$s^*[\Sigma_{1,2} \cap H^3_{P^5}] = ([H_{P^1} \times P^2] + [P^1 \times H_{P^2}])^3 = 3[H_{P^1} \times H^2_{P^2}].$$

Since $H_{P^1} \times H^2_{P^2}$ is a point of $P^1(\mathbb{C}) \times P^2(\mathbb{C})$,

$$\int_{P^4} \overline{\Sigma_{1,2}} \cap H^3_{P^4} = \int_{P^5} \Sigma_{1,2} \cap H^3_{P^5} = \int_{P^1 \times P^2} s^*[\Sigma_{1,2} \cap H^3_{P^5}] = 3,$$

i.e., $\deg \overline{\Sigma_{1,2}} = 3$. ■

By Theorem 3.1 and Theorem 3.2, we have:

**Proposition 3.4.** *We denote by $f : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^4(\mathbb{C})$ the composite of the Segre map $s : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^5(\mathbb{C})$ and a generic linear projection $\pi_p : P^5(\mathbb{C}) \to P^4(\mathbb{C})$. Concerning the multiple-point loci and the singular locus of $f$, we have the following:*

$$[M_2] = [P^1 \times H_{P^2}] \tag{3.1}$$
$$[M_3] = [M_4] = 0, \tag{3.2}$$
$$[R] = [H_{P^1} \times H_{P^2}] + [P^1 \times H^2_{P^2}] \tag{3.3}$$

**Proposition 3.5.** *Concerning the various singular loci of $\overline{X} := \overline{\Sigma}_{1,2} = f(P^1 \times P^2)$, we have the following:*

$$\deg[\overline{D}] = 1, \tag{3.4}$$
$$\deg[\overline{C}] = 2, \tag{3.5}$$
$$[\overline{T}] = [\Sigma\overline{q}] = [\Sigma\overline{s}] = 0. \tag{3.6}$$

**Proof:** Since $f_*[M_2] = 2[\overline{D}]$, by the projection formula,

$$f_*([M_2] \cdot f^*[H^2_{P^4}]) = 2[\overline{D}] \cdot [H^2_{P^4}]. \tag{3.7}$$

Since

$$f^*[H_{P^4}] = [H_{P^1} \times P^2] + [P^1 \times H_{P^2}], \tag{3.8}$$

we have

$$f^*[H_{P^4}]^2 = ([H_{P^1} \times P^2] + [P^1 \times H_{P^2}])^2 = 2[H_{P^1} \times H_{P^2}] + [P^1 \times H^2_{P^2}].$$

Hence, by (3.1)

$$\begin{aligned}[M_2]\cdot f^*[H_{P^4}]^2 &= [P^1\times H_{P^2}]\cdot(2[H_{P^1}\times H_{P^2}]+[P^1\times H^2_{P^2}])\\ &= 2[H_{P^1}\times H^2_{P^2}].\end{aligned}$$

Therefore, since $H_{P^1}\times H^2_{P^2}$ is a point of $P^1\times P^2$, by (3.7) we have

$$\int_{P^4}[\overline{D}]\cdot[H_{P^4}]^2 = 1.$$

Similarly, using the fact $f_*[R]=[\overline{C}]$, we can prove (3.5) as follows: By the projection formula,

$$f_*([R]\cdot f^*[H_{P^4}]) = [\overline{C}]\cdot[H_{P^4}]. \tag{3.9}$$

By (3.3) and (3.8),

$$\begin{aligned}[R]\cdot f^*[H_{P^4}] &= ([H_{P^1}\times H_{P^2}]+[P^1\times H^2_{P^2}])\cdot([H_{P^1}\times P^2]+[P^1\times H_{P^2}])\\ &= [H_{P^1}\times H^2_{P^2}]+[H_{P^1}\times H^2_{P^2}] = 2[H_{P^1}\times H^2_{P^2}]\end{aligned}$$

Therefore, since $H_{P^1}\times H^2_{P^2}$ is a point of $P^1\times P^2$, by (3.9),

$$\int_{P^4}[\overline{C}]\cdot[H_{P^4}] = \int_{P^1\times P^2}[R]\cdot f^*[H_{P^4}] = 2.$$

■

By Proposition 3.3, Proposition 3.4, Proposition 3.5, Proposition 2.1 and the formula for $\int_X c_3$ in Theorem 2.2, we have:

**Proposition 3.6.** *Concerning the class* $c$ *of* $\overline{X}$ *and the Euler Poincaré characteristic* $\chi(\overline{C},\mathcal{O}_{\overline{C}})$ *of the cuspidal point locus (smooth curve)* $\overline{C}$ *of* $\overline{X}$, *we have the following:*

$$c = 0, \qquad \chi(\overline{C},\mathcal{O}_{\overline{C}}) = 1.$$

The concrete equation of $\overline{\Sigma_{1,2}}$ can be calculated as follows: The Gröbner basis for the homogeneous ideal of $\Sigma_{1,2}$ in $P^5(\mathbb{C})$ is given by

$$x_0y_1 - x_1y_0,\quad x_0y_2 - x_2y_0,\quad x_1y_2 - x_2y_1.$$

Hence the point $p := [1:0:0:0:1:0]$ is not included in $\Sigma_{1,2}$. We consider the projection $\pi_p$ from the point $p$ to the hyperplane

$$H : x_0 = 0.$$

This projection $\pi_p$ is given by

$$\begin{aligned}[x_0:x_1:x_2:y_0:y_1:y_2] &\in P^5(\mathbb{C})\\ &\to (a|x)p-(a|p)x = x_0p - x = [0:x_1:x_2:y_0:y_1-x_0:y_2]\in H.\end{aligned}$$

where $a = [1:0:0:0:0:0]$ is the normal vector of the hypersurface $H$, and $(\ |\ )$ denotes the inner product. We regard $H$ as $P^4(\mathbb{C})$ and denote its homogeneous coordinates by $[z_0:z_1:z_2:z_3:z_4]$. Then $\pi_p\circ s : P^1(\mathbb{C})\times P^2(\mathbb{C})\to P^4(\mathbb{C})$ is given by

$$\begin{aligned}[s_0:s_1]\times[t_0:t_1:t_2] &\in P^1(\mathbb{C})\times P^2(\mathbb{C})\\ &\to [z_0:z_1:z_2:z_3:z_4] = [s_0t_1:s_0t_2:s_1t_0:s_1t_1-s_0t_0:s_1t_2]\in P^4(\mathbb{C}).\end{aligned}$$

We set

$$\overline{X} := (\pi_p\circ s)(P^1(\mathbb{C})\times P^2(\mathbb{C})).$$

Computing the Gröbner basis for the homogeneous ideal of $\overline{X}$ in $P^4(\mathbb{C})$ by the aid of computer, we obtain the defining equation of $\overline{X}$ as follows:

$$\overline{X}: \ z_2 z_1^2 + z_3(z_1 z_4) - z_0 z_4^2 = 0.$$

The singular loci of $\overline{X}$ are:

$$\overline{D}: \ \left\{ z_1 = z_4 = 0 \right\},$$

$$\overline{C}: \ \left\{ z_1 = z_4 = 0 \right\} \cap \left\{ z_3^2 + 4 z_0 z_2 = 0 \right\}.$$

**Example 3.2** (Generic projection of rational scroll $X_{2,2,2}$ in $P^8(\mathbb{C})$): Let $\nu_2 : P^1(\mathbb{C}) \to P^2(\mathbb{C})$ be the quadratic Veronese map (embedding), $s : P^2(\mathbb{C}) \times P^2(\mathbb{C}) \to P^8(\mathbb{C})$ the Segre map, and consider the composition

$$P^1(\mathbb{C}) \times P^2(\mathbb{C}) \xrightarrow{\nu_2 \times id} P^2(\mathbb{C}) \times P^2(\mathbb{C}) \xrightarrow{s} P^8(\mathbb{C}).$$

The image of this map is a *rational normal scroll*, and is denoted by $X_{2,2,2}$, whose meaning is as follows: We take three points $p_0, p_1, p_2$ in general position in the second factor $P^2(\mathbb{C})$, and set

$$L_0 := s(P^2(\mathbb{C}) \times p_0),$$
$$L_1 := s(P^2(\mathbb{C}) \times p_1),$$
$$L_2 := s(P^2(\mathbb{C}) \times p_2).$$

These are three 2-dimensional linear subspaces in general position in $P^8(\mathbb{C})$. Furthermore, we set

$$C_0 := (s \circ (\nu_2 \times id))(P^1(\mathbb{C}) \times p_0),$$
$$C_1 := (s \circ (\nu_2 \times id))(P^1(\mathbb{C}) \times p_1),$$
$$C_2 := (s \circ (\nu_2 \times id))(P^1(\mathbb{C}) \times p_2).$$

Each $C_i$ is a quadric in $L_i$. We denote the natural isomorphisms by

$$\varphi_i : C_0 \to C_i \quad (i = 1, 2).$$

Then $X_{2,2,2}$ is described as

$$X_{2,2,2} = \bigcup_{p \in C_0} \overline{p, \varphi_1(p), \varphi_2(p)},$$

where $\overline{p, \varphi_1(p), \varphi_2(p)}$ denotes the 2-dimensional linear subspace of $P^8(\mathbb{C})$, generated by $p, \varphi_1(p)$ and $\varphi_2(p)$. We denote by $\overline{X_{2,2,2}}$ the image of $X_{2,2,2}$ by a *generic linear projection* to a 4-dimensional linear subspace of $P^8(\mathbb{C})$. The center of this projection is a 3-dimensional linear subspace of $P^8(\mathbb{C})$. By Theorem 3.1, Theorem 3.2, Proposition 2.1, the formula for $\int_X c_3$ in Theorem 2.2 and Remark 3.1 below, we have the following concerning the degrees of $\overline{X_{2,2,2}}$ itself and the various singular loci of it:

**Proposition 3.7.**

$$\deg[\overline{X_{2,2,2}}] = 6, \quad \deg[\overline{D}] = 10, \quad \deg[\overline{T}] = 4, \quad \deg[\overline{C}] = 8, \quad \#[\Sigma\overline{q}] = 0, \quad \#[\Sigma\overline{s}] = 12,$$
$$[D] = 4[H_{P^1} \times H_{P^2} + P^1 \times H_{P^2}], \quad [T] = 6[H_{P^1} \times H_{P^2}] + 3[P^1 \times H_{P^2}^2],$$
$$[C] = 6[H_{P^1} \times H_{P^2}] + 3[P^1 \times H_{P^2}^2], \quad [\Sigma q] = 0, \quad \#[\Sigma s] = 12,$$
$$c = 0, \quad \chi(\overline{C}, \mathcal{O}_{\overline{C}}) = 1.$$

We are now going to find out the concrete equation of $\overline{X_{2,2,2}}$. We recall that the map

$$g := s \circ (\nu_2 \circ id) : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^8(\mathbb{C})$$

is defined by

$$\begin{aligned}
&[s_0 : s_1] \times [t_0 : t_1 : t_2] \in P^1(\mathbb{C}) \times P^2(\mathbb{C}) \\
&\to [s_0^2 t_0 : s_1^2 t_0 : s_0 s_1 t_0 : s_0^2 t_1 : s_1^2 t_1 : s_0 s_1 t_1 : s_0^2 t_2 : s_1^2 t_2 : s_0 s_1 t_2] \\
&\quad = [x_0 : x_1 : x_2 : y_0 : y_1 : y_2 : z_0 : z_1 : z_2] \in P^8(\mathbb{C}),
\end{aligned}$$

and $\quad X_{2,2,2} := g(P^1(\mathbb{C}) \times P^2(\mathbb{C}))$.

First, we choose a generic linear projection $\pi_{\Lambda_{(2)}} : P^8(\mathbb{C}) \to P^5(\mathbb{C})$ such that:

(i) $\quad \Lambda_{(2)} \cap X_{2,2,2} = \emptyset$,

(ii) $\quad \Lambda_{(2)} \cap \{x_0 = x_1 = x_2 = 0\} = \Lambda_{(2)} \cap \{y_0 = y_1 = y_2 = 0\} = \Lambda_{(2)} \cap \{z_0 = z_1 = z_2 = 0\} = \emptyset$.

Let $\pi_{\Lambda_{(2)}} : P^8(\mathbb{C}) \to P^5(\mathbb{C})$ be the map associated to the following matrix:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

Then the conditions (i) and (ii) are satisfied, and $f' := \pi_{\Lambda_{(2)}} \circ g : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^5(\mathbb{C})$ is given by

$$\begin{cases}
a_0 &= s_0^2 t_0 - s_0 s_1 t_1, \\
a_1 &= -s_0 s_1 t_0 + s_0^2 t_1 - s_0 s_1 t_2, \\
a_2 &= -s_0 s_1 t_0 - s_0 s_1 t_1 + s_0^2 t_2, \\
a_3 &= s_1^2 t_0 - s_0 s_1 t_2, \\
a_4 &= s_1^2 t_1 - s_0 s_1 t_1 - s_0 s_1 t_2, \\
a_5 &= -s_0 s_1 t_0 + s_1^2 t_2,
\end{cases}$$

where $[a_0 : a_1 : a_2 : a_3 : a_4 : a_5]$ is the homogeneous coordinate on $P^5(\mathbb{C})$. Computing the Gröbner basis for the homogeneous ideal of $X'_{2,2,2} := f'(P^1(\mathbb{C}) \times P^2(\mathbb{C})) \subset P^5(\mathbb{C})$, we can see that the point

$$p = [0 : 1 : 0 : 1 : 0 : 0]$$

is not included in $X'_{2,2,2}$. We consider the projection $\pi_p$ from the point $p$ to the hyperplane

$$H : (p|a) = a_1 + a_3 = 0.$$

The projection $\pi_p$ is given by

$$\begin{aligned}
a = [a_0 : a_1 : a_2 : a_3 : a_4 : a_5] &\in P^5(\mathbb{C}) \\
&\to -(p|a)p + (p|p)a = -(a_1 + a_3)p + 2a \\
&= [2a_0 : a_1 - a_3 : 2a_2 : -a_1 + a_3 : 2a_4 : 2a_5] \in H.
\end{aligned}$$

We regard $H$ as $P^4(\mathbb{C})$ and take $[a_0 : a_1 : a_2 : a_4 : a_5]$ as its homogeneous coordinate. We denote $[a_0 : a_1 : a_2 : a_4 : a_5]$ by $[b_0 : b_1 : b_2 : b_3 : b_4]$, then $\pi_p : P^5(\mathbb{C}) \to P^4(\mathbb{C})$ is given by

$$b_0 = 2a_0, \quad b_1 = a_1 - a_3, \quad b_2 = 2a_2, \\
b_3 = 2a_4, \quad b_4 = 2a_5$$

Then $f := \pi_p \circ (\pi_{\Lambda_{(2)}} \circ g) : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^4(\mathbb{C})$ is given by

$$\begin{cases} b_0 &= 2(s_0^2 t_0 - s_0 s_1 t_1), \\ b_1 &= -s_0 s_1 t_0 - s_1^2 t_0 + s_0^2 t_1, \\ b_2 &= 2(-s_0 s_1 t_0 - s_0 s_1 t_1 + s_0^2 t_2), \\ b_3 &= 2(s_1^2 t_1 - s_0 s_1 t_1 - s_0 s_1 t_2), \\ b_4 &= 2(-s_0 s_1 t_0 + s_1^2 t_2), \end{cases}$$

We set

$$\overline{X_{2,2,2}} := f(P^1(\mathbb{C}) \times P^2(\mathbb{C})).$$

Computing the Gröbner basis for the homogeneous ideal of $\overline{X_{2,2,2}}$ in $P^4(\mathbb{C})$ by the aid of computer, we obtain the defining equation of $\overline{X_{2,2,2}}$ as follows:

$$\begin{aligned}
(3.10)\quad F = {} & 11b_0^5b_3 - 14b_0^4b_1b_3 - 64b_0^3b_1^2b_3 - 40b_0^2b_1^3b_3 - 26b_0^4b_2b_3 + 4b_0^3b_1b_2b_3 \\
& +24b_0^2b_1^2b_2b_3 + 26b_0^3b_2^2b_3 - 30b_0^2b_1b_2^2b_3 - 20b_0b_1^2b_2^2b_3 - 26b_0^2b_2^3b_3 + 12b_0b_1b_2^3b_3 \\
& +11b_0b_2^4b_3 - 3b_0^4b_3^2 + 62b_0^3b_1b_3^2 + 104b_0^2b_1^2b_3^2 + 40b_0b_1^3b_3^2 + 19b_0^3b_2b_3^2 - 84b_0^2b_1b_2b_3^2 \\
& -84b_0b_1^2b_2b_3^2 + 8b_0^2b_2^2b_3^2 + 66b_0b_1b_2^2b_3^2 + 20b_1^2b_2^2b_3^2 - 33b_0b_2^3b_3^2 - 32b_1b_2^3b_3^2 \\
& +11b_2^4b_3^2 - 11b_0^3b_3^3 - 34b_0^2b_1b_3^3 - 20b_0b_1^2b_3^3 + 11b_0^2b_2b_3^3 + 34b_0b_1b_2b_3^3 + 20b_1^2b_2b_3^3 \\
& +8b_0b_2^2b_3^3 - 2b_1b_2^2b_3^3 - 7b_2^3b_3^3 + 4b_0^2b_3^4 - 2b_0b_1b_3^4 - 8b_0b_2b_3^4 + 4b_2^2b_3^4 + b_0b_3^5 - b_2b_3^5 \\
& -22b_0^4b_1b_4 + 28b_0^3b_1^2b_4 + 128b_0^2b_1^3b_4 + 80b_0b_1^4b_4 - 11b_0^4b_2b_4 + 66b_0^3b_1b_2b_4 \\
& +56b_0^2b_1^2b_2b_4 - 8b_0b_1^3b_2b_4 + 26b_0^3b_2^2b_4 - 56b_0^2b_1b_2^2b_4 + 36b_0b_1^2b_2^2b_4 + 40b_1^3b_2^2b_4 \\
& -26b_0^2b_2^3b_4 + 82b_0b_1b_2^3b_4 - 4b_1^2b_2^3b_4 + 26b_0b_2^4b_4 - 34b_1b_2^4b_4 - 11b_2^5b_4 + 8b_0^4b_3b_4 \\
& -20b_0^3b_1b_3b_4 - 112b_0^2b_1^2b_3b_4 - 80b_0b_1^3b_3b_4 + 11b_0^3b_2b_3b_4 - 62b_0^2b_1b_2b_3b_4 \\
& -8b_0b_1^2b_2b_3b_4 + 40b_1^3b_2b_3b_4 - 45b_0^2b_2^2b_3b_4 + 2b_0b_1b_2^2b_3b_4 + 36b_1^2b_2^2b_3b_4 \\
& +9b_0b_2^3b_3b_4 + 22b_1b_2^3b_3b_4 + 7b_2^4b_3b_4 - 9b_0^3b_3^2b_4 + 82b_0^2b_1b_3^2b_4 + 92b_0b_1^2b_3^2b_4 \\
& +31b_0^2b_2b_3^2b_4 - 78b_0b_1b_2b_3^2b_4 - 20b_1^2b_2b_3^2b_4 + 23b_0b_2^2b_3^2b_4 + 28b_1b_2^2b_3^2b_4 \\
& -35b_2^3b_3^2b_4 - 9b_0^2b_3^3b_4 - 52b_0b_1b_3^3b_4 - 5b_0b_2b_3^3b_4 + 36b_1b_2b_3^3b_4 + 15b_2^2b_3^3b_4 \\
& +8b_0b_3^4b_4 - 8b_2b_3^4b_4 + 22b_0^4b_4^2 - 44b_0^3b_1b_4^2 - 120b_0^2b_1^2b_4^2 - 48b_0b_1^3b_4^2 - 60b_0^3b_2b_4^2 \\
& +18b_0^2b_1b_2b_4^2 + 64b_0b_1^2b_2b_4^2 + 55b_0^2b_2^2b_4^2 - 60b_0b_1b_2^2b_4^2 - 4b_1^2b_2^2b_4^2 - 52b_0b_2^3b_4^2 \\
& +60b_1b_2^3b_4^2 + 31b_2^4b_4^2 - 13b_0^3b_3b_4^2 + 110b_0^2b_1b_3b_4^2 + 112b_0b_1^2b_3b_4^2 + 79b_0^2b_2b_3b_4^2 \\
& -128b_0b_1b_2b_3b_4^2 - 24b_1^2b_2b_3b_4^2 - 48b_0b_2^2b_3b_4^2 + 40b_1b_2^2b_3b_4^2 - 14b_2^3b_3b_4^2 \\
& -15b_0^2b_3^2b_4^2 - 66b_0b_1b_3^2b_4^2 - 2b_0b_2b_3^2b_4^2 + 44b_1b_2b_3^2b_4^2 + 25b_2^2b_3^2b_4^2 + 11b_0b_3^3b_4^2 \\
& -11b_2b_3^3b_4^2 + 5b_0^3b_4^3 + 20b_0^2b_1b_4^3 + 20b_0b_1^2b_4^3 + 5b_0^2b_2b_4^3 - 30b_0b_1b_2b_4^3 + 2b_0b_2^2b_4^3 \\
& -2b_1b_2^2b_4^3 - 17b_2^3b_4^3 - 10b_0^2b_3b_4^3 - 20b_0b_1b_3b_4^3 + 10b_0b_2b_3b_4^3 + 10b_1b_2b_3b_4^3 \\
& +11b_2^2b_3b_4^3 + 5b_0b_3^2b_4^3 - 5b_2b_3^2b_4^3 + 5b_2^2b_4^4.
\end{aligned}$$

( 134 terms, compared with ${}_{n-1+d}C_d = {}_{10}C_6 = 210$)

In order to obtain the generators of the ideal for the cuspidal point locus C of the map $f := \pi_p \circ (\pi_{\Lambda_{(2)}} \circ g) : P^1(\mathbb{C}) \times P^2(\mathbb{C}) \to P^4(\mathbb{C})$, we compute all 4-minors $m_1, \cdots, m_{25}$ of the Jacobian matrix

$$\frac{\partial(b_0, b_1, b_2, b_3, b_4)}{\partial(s_0, s_1, t_0, t_1, t_2)}$$

of the map f. Among the 4-minors, there exist

$$\begin{aligned}
m_5 &= 8s_0^3\,(2s_0^3s_1t_0 + 2s_0^2s_1^2t_0 + s_1^4t_0 + s_0^4t_1 - s_0^3s_1t_1 + s_0^2s_1^2t_1 + s_0s_1^3t_1 + s_1^4t_1 + s_0^4t_2 - s_0^2s_1^2t_2 - s_0s_1^3t_2) \\
m_{10} &= 8s_0^2\,(s_0^5t_0 - s_0^3s_1^2t_0 - 2s_0s_1^4t_0 - s_1^5t_0 + s_0^4s_1t_1 - s_0^3s_1^2t_1 - s_0^2s_1^3t_1 - 2s_0^4s_1t_2 + 2s_0^2s_1^3t_2 + 2s_0s_1^4t_2)
\end{aligned}$$

Note that thses expressions are linear with respect $t_0, t_1, t_2$. If we put $\lambda = s_1/s_0$, then

$$
\begin{aligned}
\frac{m_5}{8s_0^7} &= (2\lambda + 2\lambda^2 + \lambda^4)t_0 + (1 - \lambda + \lambda^2 + \lambda^3 + \lambda^4)t_1 + (1 - \lambda^2 - \lambda^3)t_2, \\
\frac{m_{10}}{8s_0^7} &= (1 - \lambda^2 - 2\lambda^4 - \lambda^5)t_0 + (\lambda - \lambda^2 - \lambda^3)t_1 - (2\lambda - 2\lambda^3 - 2\lambda^4)t_2.
\end{aligned}
$$

We solve these simultaneous linear equations with respect to $t_0, t_1, t_2$, then we have

(3.11) $[t_0 : t_1 : t_2]$

$$
\begin{aligned}
&= [\lambda\left(3 - 3\lambda + \lambda^2 + 2\lambda^3 + 2\lambda^4\right) : -\left(1 + 3\lambda^2 + 4\lambda^3 - 2\lambda^4 + \lambda^5\right) : 1 - \lambda - \lambda^2 + 2\lambda^3 + 2\lambda^5 + \lambda^6] \\
&= [\mu(3\mu^5 - 3\mu^4 + \mu^3 + 2\mu^2 + 2\mu) : -\mu(\mu\,5 + 3\mu^3 + 4\mu^2 - 2\mu + 1 : \mu^6 - \mu^5 - \mu^4 + 2\mu^3 + 2\mu + 1],
\end{aligned}
$$

where $\mu = 1/\lambda$. Substituting (3.11) to all the 4-minors $m_i$, $1 \le i \le 25$, we can make sure that (3.11) is a parametric representation of the cuspidal point locus C of the map f. Thus C is a non-singular rational curve, and so $\chi(\overline{C}, \mathcal{O}_{\overline{C}}) = 1$.

The generators of the ideal for the singular subscheme $\bar{J}$ of $\overline{X_{2,2,2}}$ are

$$
\frac{\partial F}{\partial b_0}, \quad \frac{\partial F}{\partial b_1}, \quad \ldots, \quad \frac{\partial F}{\partial b_4}.
$$

Pulling back these by the map f, we obtain the generators for the ideal of the scheme theoretic inverse J of $\bar{J}$ by f. From the fact that $\mathcal{J}_J = \mathcal{J}_D \otimes_{\mathcal{J}_X} \mathcal{J}_C$, where $X = P^1(\mathbb{C}) \times P^2(\mathbb{C})$, and $\mathcal{J}_J$, $\mathcal{J}_D$, $\mathcal{J}_C$ are the ideal sheaves of J, D and C, respectively, it follows that the equation G of the double point locus D of f is defined by the following equation:

$$
\begin{aligned}
G \;=\; & 11s_0^4t_0^4 + 25s_0^3s_1t_0^4 + 18s_0^2s_1^2t_0^4 + 5s_0s_1^3t_0^4 - 7s_0^4t_0^3t_1 + 6s_0^3s_1t_0^3t_1 + 2s_0^2s_1^2t_0^3t_1 \\
& +5s_0s_1^3t_0^3t_1 - 16s_0^4t_0^2t_1^2 + 8s_0^3s_1t_0^2t_1^2 - 15s_0^2s_1^2t_0^2t_1^2 + 17s_0s_1^3t_0^2t_1^2 + 5s_1^4t_0^2t_1^2 - 5s_0^4t_0t_1^3 \\
& +16s_0^3s_1t_0t_1^3 - 24s_0^2s_1^2t_0t_1^3 + 13s_0s_1^3t_0t_1^3 + 5s_0^3s_1t_1^4 - 10s_0^2s_1^2t_1^4 \\
& +6s_0s_1^3t_1^4 - s_1^4t_1^4 - 26s_0^4t_0^3t_2 - 37s_0^3s_1t_0^3t_2 - 32s_0^2s_1^2t_0^3t_2 \\
& -12s_0s_1^3t_0^3t_2 - 5s_1^4t_0^3t_2 + 2s_0^4t_0^2t_1t_2 + 13s_0^3s_1t_0^2t_1t_2 - 48s_0^2s_1^2t_0^2t_1t_2 \\
& +5s_0s_1^3t_0^2t_1t_2 - 5s_1^4t_0^2t_1t_2 + 6s_0^4t_0t_1^2t_2 + 34s_0^3s_1t_0t_1^2t_2 \\
& -49s_0^2s_1^2t_0t_1^2t_2 + 19s_0s_1^3t_0t_1^2t_2 - 18s_1^4t_0t_1^2t_2 + 9s_0^3s_1t_1^3t_2 \\
& -21s_0^2s_1^2t_1^3t_2 + 21s_0s_1^3t_1^3t_2 - 8s_1^4t_1^3t_2 + 26s_0^4t_0^2t_2^2 + 48s_0^3s_1t_0^2t_2^2 \\
& +14s_0^2s_1^2t_0^2t_2^2 + 12s_0s_1^3t_0^2t_2^2 - 6s_1^4t_0^2t_2^2 - 15s_0^4t_0t_1t_2^2 + 37s_0^3s_1t_0t_1t_2^2 \\
& +s_0^2s_1^2t_0t_1t_2^2 + 35s_0s_1^3t_0t_1t_2^2 - 22s_1^4t_0t_1t_2^2 - 26s_0^4t_0t_2^3 - 23s_0^3s_1t_0t_2^3 \\
& -5s_0^4t_1^2t_2^2 + 6s_0^3s_1t_1^2t_2^2 - 24s_0^2s_1^2t_1^2t_2^2 + 36s_0s_1^3t_1^2t_2^2 - 11s_1^4t_1^2t_2^2 \\
& -9s_0^2s_1^2t_0t_2^3 + 22s_0s_1^3t_0t_2^3 - 5s_1^4t_0t_2^3 + 6s_0^4t_1t_2^3 - 7s_0^3s_1t_1t_2^3 - 30s_0^2s_1^2t_1t_2^3 \\
& +27s_0s_1^3t_1t_2^3 - 5s_1^4t_1t_2^3 + 11s_0^4t_2^4 + 7s_0^3s_1t_2^4 - 16s_0^2s_1^2t_2^4 + 5s_0s_1^3t_2^4.
\end{aligned}
$$

(69 terms, compared with ${}_5C_4 \times_6 C_4 = 5 \times 15 = 75$)

Since the triple point locus T of f is nothing but the singular locus of D, the generators of the ideal for the triple point locus C of f are generated by

(3.12)
$$
\frac{\partial G}{\partial s_0}, \quad \frac{\partial G}{\partial s_1}, \quad \frac{\partial G}{\partial t_0}, \quad \frac{\partial G}{\partial t_1}, \quad \frac{\partial G}{\partial t_2}.
$$

In order to obtain the stationary point locus $\sum s$ of f, we substitute the parametric representation of the cuspidal point curve C in (3.11) into (3.12) since $\sum s = C \cap T$, equate these to zero, and solve them by the aid of computer. Then it turns out that the stationary point locus $\sum s$ of f consists of the 12 points corresponding to the roots of following equation in $\lambda$:

$$
5\lambda^{12} + 10\lambda^{11} + 8\lambda^{10} - 24\lambda^9 - 76\lambda^8 - 48\lambda^7 + 155\lambda^6 + 34\lambda^5 - 259\lambda^4 + 32\lambda^3 + 115\lambda^2 - 68\lambda + 8 = 0.
$$

**Example 3.3**(Generic projection of the image of $P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ in $P^7(\mathbb{C})$ by the Segre map): Let $s : P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times P^1(\mathbb{C}) \to P^7(\mathbb{C})$ be the map defined by

$$\begin{aligned}[s_0 : s_1] \times [t_0 : t_1] \times [u_0 : u_1] &\in P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times P^1(\mathbb{C}) \\ &\to [s_0t_0u_0 : s_0t_0u_1 : s_0t_1u_0 : s_0t_1u_1 : s_1t_0u_0 : s_1t_0u_1 : s_1t_1u_0 : s_1t_1u_1] \\ &\quad = [x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3] \in P^7(\mathbb{C})\end{aligned}$$

i.e., the *Segre map* from $P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times P^1(\mathbb{C})$ to $P^7(\mathbb{C})$. We set

$$\Sigma_{1,1,1} := s(P^1(\mathbb{C}) \times P^1(\mathbb{C}) \times P^1(\mathbb{C})).$$

We denote by $\overline{\Sigma_{1,1,1}}$ the image of $\Sigma_{1,1,1}$ by a *generic linear projection* to a 4-dimensional linear subspace of $P^7(\mathbb{C})$. The center of this projection is a 2-dimensional linear subspace of $P^7(\mathbb{C})$. By the same way to prove Proposition 3.7, we have the following concerning the degrees of $\overline{\Sigma_{1,1,1}}$ itself and various singular loci of it:

**Proposition 3.8.**

$$\begin{aligned}&\deg[\overline{\Sigma_{1,1,1}}] = 6, \quad \deg[\overline{D}] = 9, \quad \deg[\overline{T}] = 4, \quad \deg[\overline{C}] = 12, \quad \#[\Sigma\overline{q}] = 1, \quad \#[\Sigma\overline{s}] = 16, \\ &[D] = 3[H_{P^1} \times P^1 \times P^1 + P^1 \times H_{P^1} \times P^1 + P^1 \times P^1 \times H_{P^1}], \\ &[T] = 4[H_{P^1} \times H_{P^1} \times P^1 + P^1 \times H_{P^1} \times H_{P^1} + H_{P^1} \times P^1 \times H_{P^1}], \\ &[C] = 4[H_{P^1} \times H_{P^1} \times P^1 + P^1 \times H_{P^1} \times H_{P^1} + H_{P^1} \times P^1 \times H_{P^1}], \\ &\#[\Sigma q] = 4, \quad \#[\Sigma s] = 16, \\ &c = 4, \quad \chi(\overline{C}, \mathcal{O}_{\overline{C}}) = 0.\end{aligned}$$

This example might be interesting, because a quadruple point exists.

**Example 3.4** (Steiner threefold): Let $\nu_2 : P^3(\mathbb{C}) \to P^9(\mathbb{C})$ be the map defined by

$$\begin{aligned}[\xi_0 : \xi_1 : \xi_2 : \xi_3] &\in P^3(\mathbb{C}) \\ &\to [\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_3^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_0\xi_3 : \xi_1\xi_2 : \xi_1\xi_3 : \xi_2\xi_3] \\ &\quad = [x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3 : y_4 : y_5] \in P^9(\mathbb{C}),\end{aligned}$$

i.e., the quadratic Veronese map (embedding). We set

$$X := \nu_2(P^3(\mathbb{C})).$$

We denote by $\overline{X}$ the image of X by a *generic linear projection* to a 4-dimensional linear subspace of $P^9(\mathbb{C})$, and call it *Steiner threefold*. The center of this projection is a 4-dimensional linear subspace of $P^9(\mathbb{C})$. By the same way to prove Proposition 3.7, we have the following concerning the degrees of the Steiner threefold itself and various singular loci of it:

**Proposition 3.9.**

$$\begin{aligned}&\deg[\overline{X}] = 8, \quad \deg[\overline{D}] = 20, \quad \deg[\overline{T}] = 20, \quad \deg[\overline{C}] = 20, \quad \#[\Sigma\overline{q}] = 5, \quad \#[\Sigma\overline{s}] = 40, \\ &\deg[D] = 10, \quad \deg[T] = 30, \quad \deg[C] = 10, \quad \#[\Sigma q] = 20, \quad \#[\Sigma s] = 40, \\ &c = 4, \quad \chi(\overline{C}, \mathcal{O}_{\overline{C}}) = -10.\end{aligned}$$

**Remark 3.1.** The number of stationary points $\Sigma\overline{s}$ in Proposition 3.7, Proposition 3.8 and Proposition 3.9 can be calculated by the identity

$$f^*[\overline{T}] = f^*[\overline{X}] \cdot T - [D] \cdot [T] - [\Sigma s] + [\Sigma q]$$

in Proposition 1.12 in [17].

We have not yet succeeded in calculating the concrete equations for $\overline{\Sigma_{1,1,1}}$ in Example 3.3 and $\overline{X}$ in Example 3.4. It sometimes happens that we obtain the equations of 3-dimensional hypersurfaces in $P^4(\mathbb{C})$, which have other kinds of multiple-points than *ordinary* and *stationary* quadruple points.

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